Cramer-Rao Bounds for Distributed System Size Estimation Using Consensus Algorithms

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INTRODUCTION AND MOTIVATION

- Wireless sensor networks (WSNs) with no fusion center
- Estimate the system size (total number of sensors) of WSNs
- Applications: network maintenance, detect nodes join or leave the network

PROBLEM STATEMENT

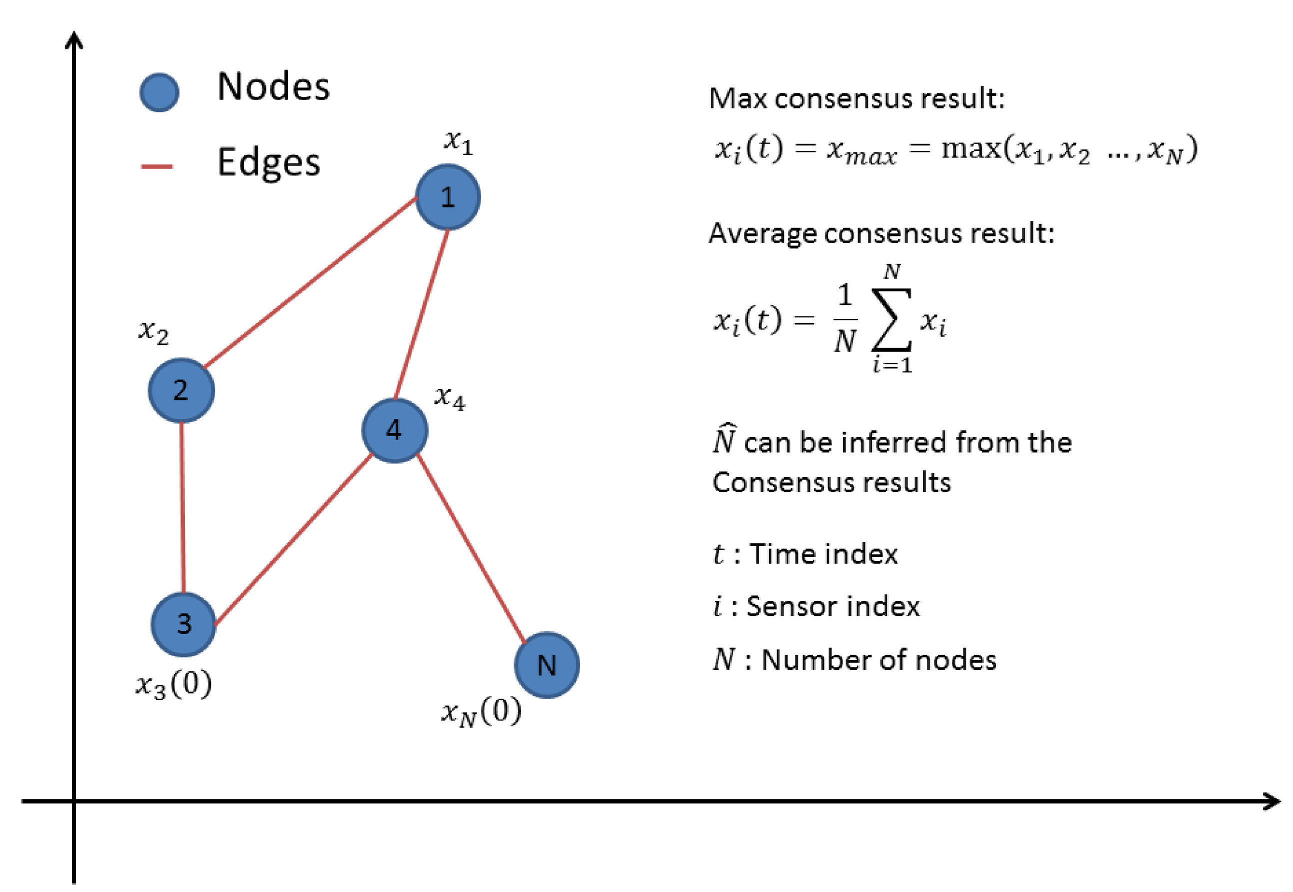
- Count number of nodes using consensus
- Random initial values, x_i generated at nodes
- Different consensus algorithms:
  - Max consensus

$$x_i(t+1) = \max\left\{x_i(t), \max_{j\in\mathbb{N}_i} x_j(t)\right\}$$

  - Average consensus

$$x_i(t+1) = W_{ii}x_i(t) + \sum_{j\in\mathbb{N}_i} W_{ij}x_j(t)$$

- System size inferred from the consensus results
- Problem: performance is affected by:
  - Types of consensus algorithm
  - Initial values, x_i
- Performance analysis: Fisher information (FI) and Cramer-Rao bounds (CRBs)



RESULTS: FI AND CRBS UNDER DIFFERENT CASES

Max Consensus in the Absence of Noise

Theorem

Assume the initial values at nodes x_i are i.i.d. with PDF $f(x)$ and CDF $F(x)$, and $f(x)$ is differentiable. When max consensus is used, the Fisher information for estimate of system size N is,

$$\mathcal{I}_{\max} = \frac{1}{N^2}.$$

The CRB is the inverse of the Fisher information, and a lower bound on the estimation variance can be expressed as

$$\mathrm{Var}\left[\hat{N}\right] \geq N^2.$$

The distribution of the initial values at nodes does not affect the Fisher information and CRB.

Average Consensus in the Absence of Noise

Theorem

Assume the initial values at nodes x_i are i.i.d. with mean μ and variance σ^2. Also assume that N is large. When average consensus is used, the Fisher information for estimate of system size N is

$$\mathcal{I}_{\text{avg}} = \frac{1}{2N^2}.$$

The CRB is the inverse of the Fisher information, and an lower bound on the estimation variance can be expressed as

$$\mathrm{Var}\left[\hat{N}\right] \geq 2N^2.$$

Max Consensus with Noise

Theorem

Assume the initial values at nodes x_i have exponential tail and its tail PDF $\lambda e^{-\lambda x}$. The distribution of the max of the initial values can be approximated using Gumbel distribution. Assume that the final error at nodes is Gaussian distributed $e \sim \mathcal{N}(\mu_e, \sigma_e^2)$. The Fisher information for estimate of system size N is bounded by

$$\mathcal{I}_{\text{nmax}} \leq \left(\frac{1}{N^2}\right)\left(\frac{\lambda^{-2}}{\sigma_e^2 + \lambda^{-2}}\right).$$

The CRB is the inverse of the Fisher information, and a lower bound on the estimation variance can be expressed as

$$\mathrm{Var}\left[\hat{N}\right] \geq N^2\left(\sigma_e^2\lambda^2 + 1\right).$$

Average Consensus with Noise

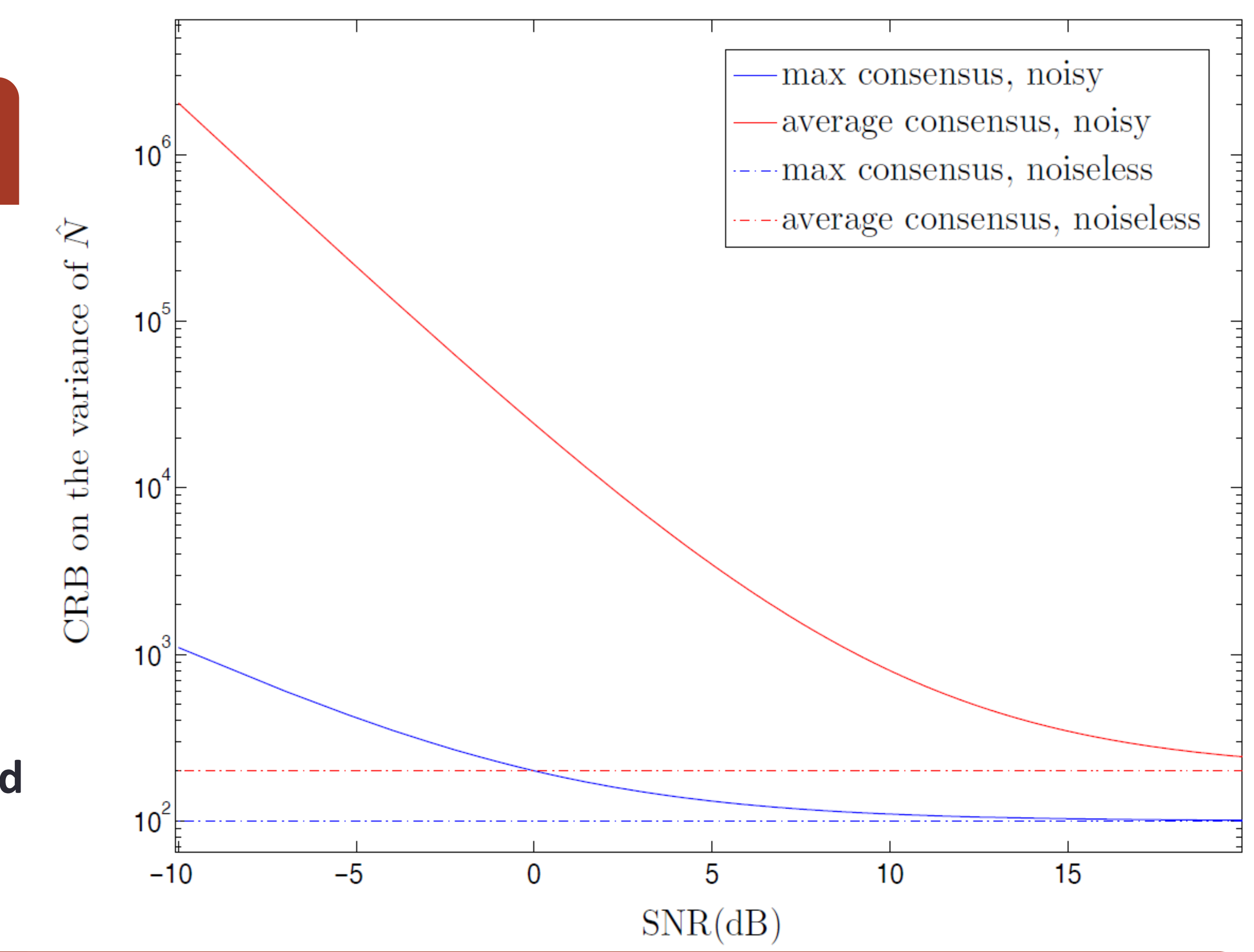
Theorem

Assume the initial values at nodes x_i are i.i.d. with mean μ and variance σ^2, the final error at nodes is Gaussian distributed $e \sim \mathcal{N}(0, \sigma_e^2)$. Also assume that N is large. When average consensus is used, the Fisher information for estimate of system size N is

$$\mathcal{I}_{\text{avg}} = \frac{1}{2N^4}\left(\frac{\sigma^4}{\left(\frac{\sigma^2}{N} + \sigma_e^2\right)^2}\right).$$

When N is large, $\mathcal{I}_{\text{avg}} \approx \frac{1}{2N^4}\left(\frac{\sigma^2}{\sigma_e^2}\right)^2 = \frac{1}{2N^4}(\text{SNR})^2$, where SNR is defined as $\text{SNR} = \frac{\sigma^2}{\sigma_e^2}$. The CRB is the inverse of the Fisher information, and an lower bound on the estimation variance can be expressed as

$$\mathrm{Var}\left[\hat{N}\right] \geq 2N^4\frac{\left(\frac{\sigma^2}{N} + \sigma_e^2\right)^2}{\sigma^4}.$$



CONCLUSIONS

- For max consensus without noise, x_i does not affect Fisher information and Cramer-Rao Bound
- Max consensus has lower CRB (noiseless case)
- How noise affect FI and CRB is presented

REFERENCES

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