Polynomial Root-MUSIC Algorithm for Efficient Broadband Direction Of Arrival Estimation

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Introduction

The popular MUSIC algorithm was recently extended to broadband scenarios via polynomial matrices and the polynomial eigenvalue decomposition. Due to the heuristic search stage, this algorithm is expensive in its computation, thus the motivation for extending the Root-MUSIC algorithm to broadband scenarios.
Aim: Propose a method for efficient direction of arrival estimation of broadband signals via polynomial matrix methods

Broadband Data Model

The signal received at the antenna array is modelled as the superposition of P steered sources, embedded in additive white Gaussian noise. Unlike narrowband sources, the time delay between antenna elements **cannot be approximated by a simple phase shift** of the carrier, but requires a linear phase shift across the frequency band. This leads to the following convolutive mixture model.

$$\boldsymbol{x}(n) = \sum_{p=1}^{P} \boldsymbol{a_p} \otimes s_p(n) + \boldsymbol{\nu}(n)$$

where a_p is the broadband steering vector, which is a vector of **fractional delay FIR filters**. When represented in terms of the z-transform, this yields the Vandermonde structured polynomial steering vector. This Vandermonde structure indicates **translational invariance** across the array

$$\boldsymbol{a_p}(z) = \begin{bmatrix} \psi_p^0(z) \\ \psi_p^1(z) \\ \vdots \\ \psi_p^{(M-1)}(z) \end{bmatrix}$$

Space-Time Covariance Matrix & PEVD

In narrowband array processing, it is common to form a spatial-only covariance matrix. For broadband sources, a **range of temporal correlations** are also of interest. This leads to the definition of the **polynomial space-time covariance matrix**.

$$\boldsymbol{R_{xx}}(z) = \sum_{\tau=-\infty}^{\infty} \boldsymbol{R_{xx}}(\tau) z^{-\tau} \qquad \boldsymbol{R_{xx}}(\tau) = E[\boldsymbol{x}(n)\boldsymbol{x}^H(n-\tau)]$$

The space time covariance matrix may also be expressed as:

$$\boldsymbol{R_{xx}}(z) = \boldsymbol{A}(z)\boldsymbol{R_{ss}}(z)\tilde{\boldsymbol{A}}(z) + \sigma_\nu^2 \boldsymbol{I}$$

Where $\boldsymbol{R_{ss}}(z)$ is the source cross spectral density matrix. The above polynomial space time covariance matrix can be decomposed into polynomial eigenvalues $\boldsymbol{\Lambda}(z)$ and eigenvectors $\boldsymbol{U}(z)$ using a **PEVD algorithm** such as SBR2, or SMD. [1]

$$\boldsymbol{R_{xx}}(z) = \boldsymbol{U}(z)\boldsymbol{\Lambda}(z)\tilde{\boldsymbol{U}}(z) = \begin{bmatrix}\boldsymbol{U_s}(z) & \boldsymbol{U_n}(z)\end{bmatrix} \begin{bmatrix}\boldsymbol{\Lambda_s}(z) & \\ & \boldsymbol{\Lambda_n}(z)\end{bmatrix} \begin{bmatrix}\tilde{\boldsymbol{U}}_s(z) \\ \tilde{\boldsymbol{U}}_n(z)\end{bmatrix}$$

These polynomial eigenvalues are representative of a **power spectral density** and can thus be evaluated for $z = e^{j\Omega}$ to determine the dimensions of the signal and noise subspaces.

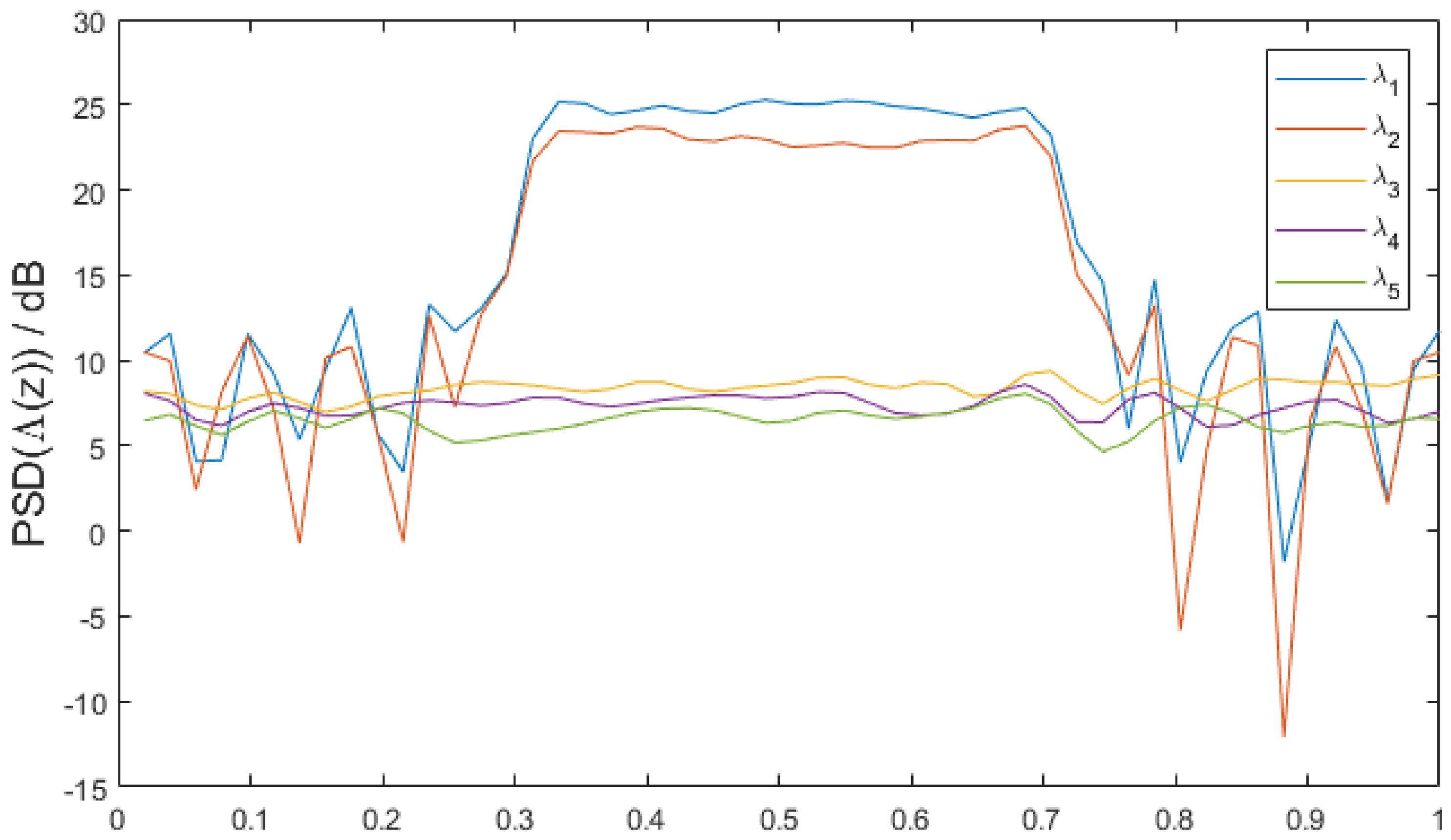

Figure: Eigenvalue Power Spectral Density

The above figure shows there are **two uncorrelated sources** in the band $\Omega \in [0.3\pi \ 0.7\pi]$

Polynomial MUSIC & Polynomial Root-MUSIC

Similarly to narrowband MUSIC, the generalised quantity, $\Gamma(z)$, can be formed: [2]

$$\Gamma(z) = \tilde{\boldsymbol{a}}_{\boldsymbol{\theta}}(z)\boldsymbol{U_n}(z)\tilde{\boldsymbol{U}}_{\boldsymbol{n}}(z)\boldsymbol{a_\theta}(z)$$

The conventional polynomial MUSIC algorithm solves the above equation by scanning all possible polynomial steering vectors across the noise subspace, i.e.

$$P_{mu}(\theta,\Omega) = \frac{1}{\tilde{\boldsymbol{a}}_{\boldsymbol{\theta}}(z)\boldsymbol{U_n}(z)\tilde{\boldsymbol{U}}_{\boldsymbol{n}}(z)\boldsymbol{a_\theta}(z)} |z = e^{j\Omega}$$

This **heuristic search is computationally expensive** due the numerous polynomial multiplications required. The polynomial Root-MUSIC algorithm aims to eliminate this heuristic search by reformulating the problem; into **solving the roots of a polynomial**.

$$\Gamma(\psi(z)) = \sum_{l=-(M-1)}^{M-1} b_l(z)\psi^l(z) \qquad \Gamma(\psi(e^{j\Omega})) = \sum_{l=-(M-1)}^{M-1} b_l(e^{j\Omega})\psi^l(e^{j\Omega})$$

Where $\boldsymbol{b_l}(z)$ is the sum of the l^{th} subdiagonal of $\boldsymbol{U_n}(z)\tilde{\boldsymbol{U}}_{\boldsymbol{n}}(z)$. The below figures demonstrate the estimated direction of arrival for sources with a ground truth DoA of -40° and 30° with frequency content in the band $\Omega \in [0.3\pi \ 0.7\pi]$.

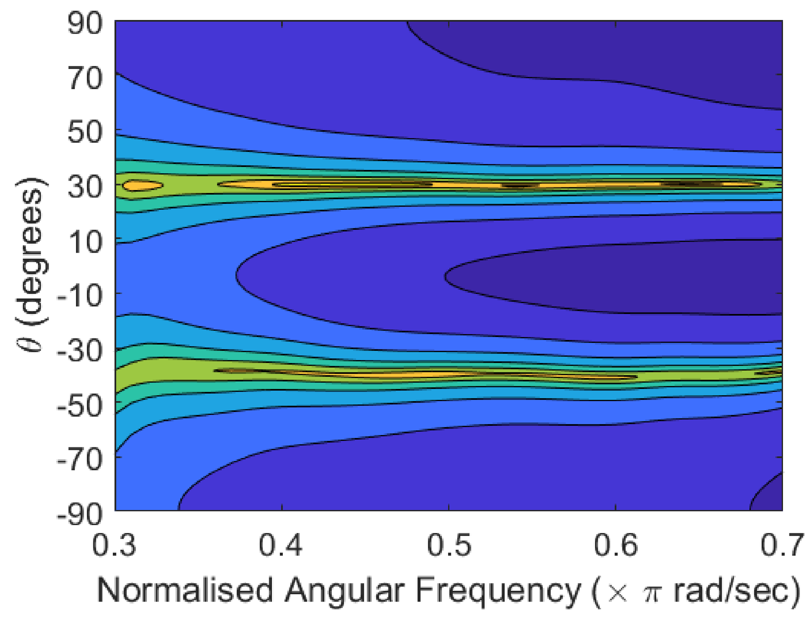

Figure: Conventional SSP MUSIC

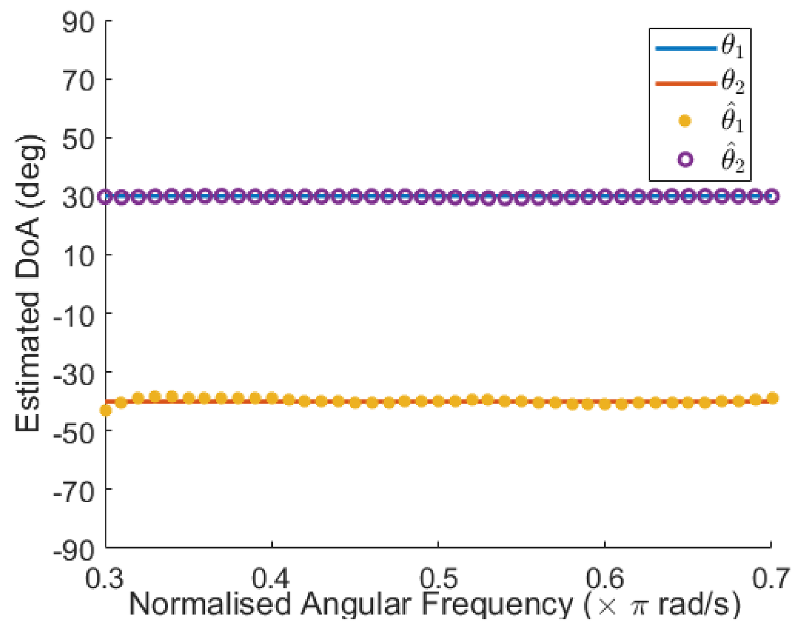

Figure: Root SSP-MUSIC

Performance Analysis

To compare the performance of the conventional Polynomial MUSIC and Polynomial Root-MUSIC at a range of SNRs, a Monte Carlo simulation was performed.

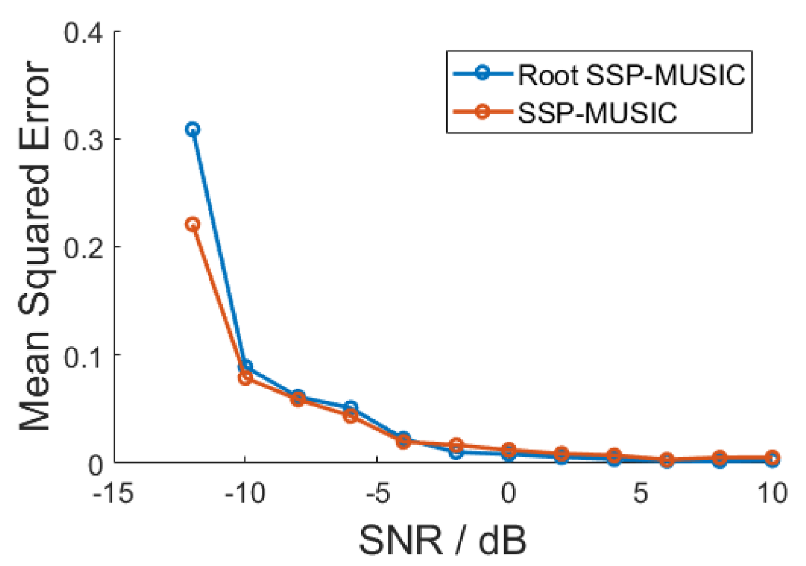

Figure: Monte Carlo Run for a range of SNRs

Table: Monte Carlo Simulation Parameters

Number of sources	1
Number of iterations	100
Number of Snapshots	500
Source DoA θ	$\in [-50^\circ \ 50^\circ]$
Source Frequency, Ω	$\in [0 \ \pi]$

In addition to SNR performance, the **computation time was analysed** and compared. As demonstrated in the table below, the mean normalised computation time was **significantly lower** for the Root SSP-MUSIC algorithm.

Table: Normalised Mean Computation time for SSP-MUSIC and Root SSP-MUSIC algorithm

N_f	30	60	120	240
Mean Norm. SSP-MUSIC Comp Time	1	1.04	1.124	1.268
Mean Norm. Root SSP-MUSIC Comp Time	0.039	0.054	0.079	0.131

Conclusions & Future Plans

A potential solution for reducing the computation time for direction of arrival estimation of broadband sources has been presented through the use of polynomial matrix methods. The performance and computation time have been analysed and compared. This research shows that the polynomial Root-MUSIC can yield similar performance to the conventional with a greatly reduced computational cost.

1. J G. McWhirter, et. al., *"An EVD algorithm for para-hermitian polynomial matrices,"* IEEE Trans. Signal Processing May 2007.
2. M. A. Alrmah, et. al., *"An extension of the music algorithm to broadband scenarios using a polynomial eigenvalue decomposition,"* EUSIPCO, Aug 2011

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