Optimal Bernoulli point estimation with applications

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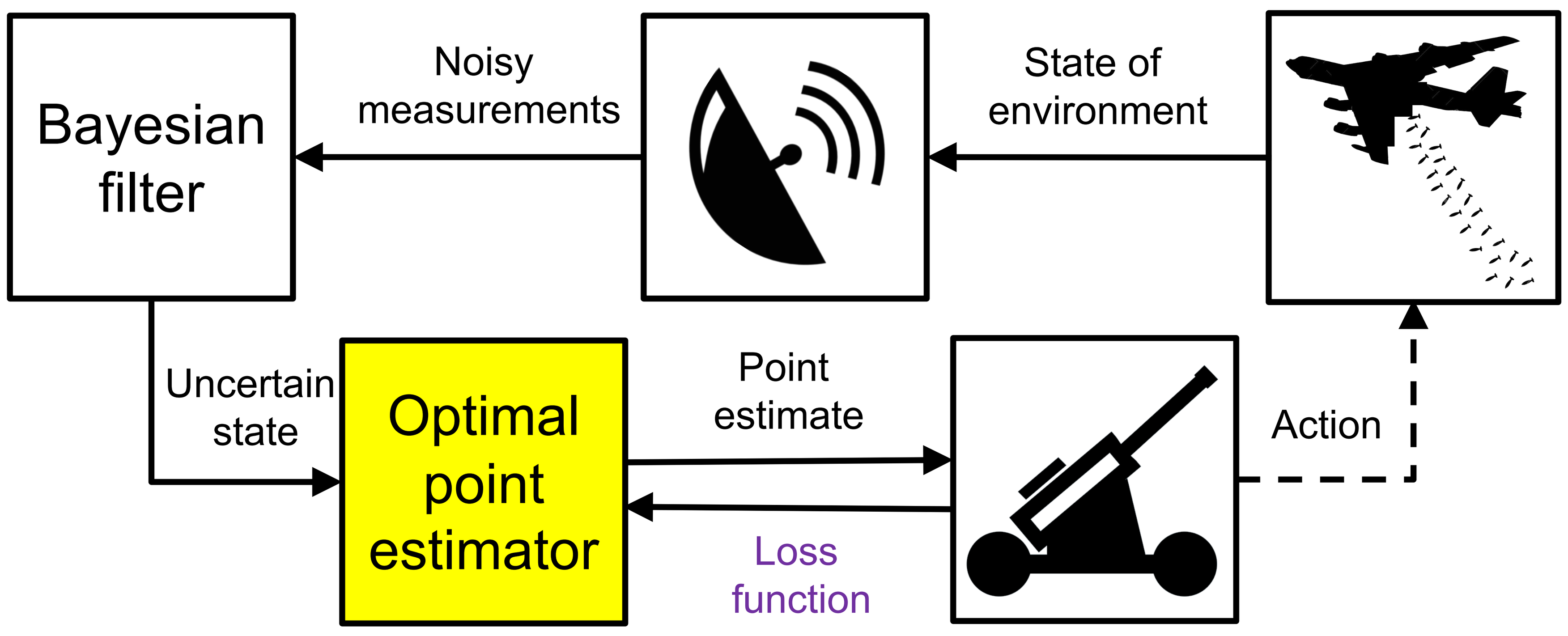
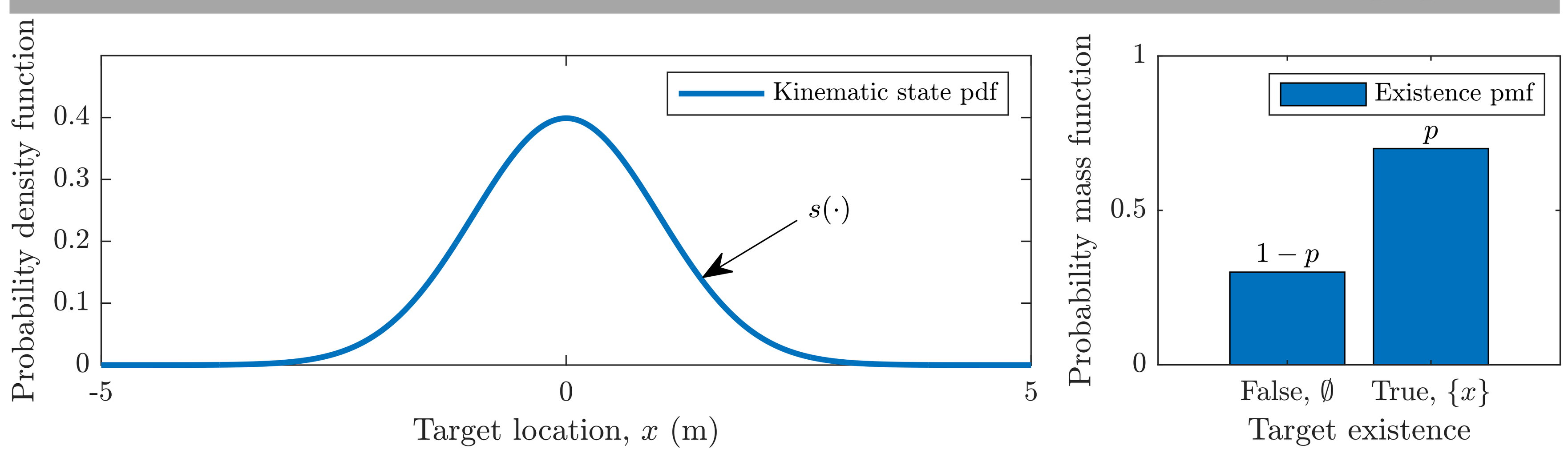
INTRODUCTION AND PROBLEM STATEMENT

APPLICATION-ORIENTED OPTIMAL POINT ESTIMATION



- Estimator removes uncertainty in the state of environment
- The estimate directs action, which is potentially fallible
- In application, the best estimate minimises the expected loss

POINT ESTIMATION WITH A BERNOULLI FILTER [1]



- A single target whose existence and location are unknown
- This **conventional estimator** is not optimal, $\Gamma = 0.5$:

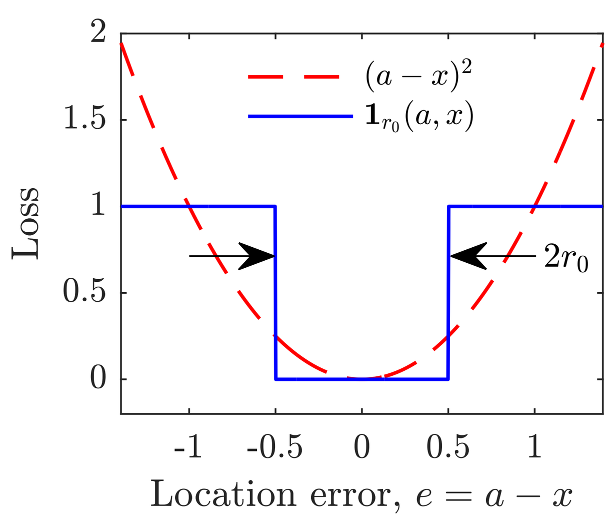
$$\hat{\alpha} = \begin{cases} \left\{\operatorname*{argmin}_{a \in \mathcal{X}} \int (a-x)^2 s(x)dx\right\}, & \text{if } p > \Gamma, \\ \emptyset, & \text{if } p < \Gamma. \end{cases}$$

- **Estimators based on OSPA-like distances** are optimal:
  - OSPA jointly quantifies errors in cardinality and location
  - Does not characterise losses in any specific application

APPLICATION-ORIENTED BERNOULLI ESTIMATOR

- Loss function combines the cost matrix and the uniform cost loss

$L(\alpha,\varphi)$	$\varphi = \emptyset$	$\varphi = \{x\}$
$\alpha = \emptyset$	c_{00}	c_{01}
$\alpha = \{a\}$	c_{10}	$c_{11} + c \cdot \mathbf{1}_{r_0}(a,x)$



- Optimal point estimate α^* and the minimum expected loss ρ^*

$$(\alpha^*, \rho^*) = \begin{cases} \left(\{a^*_{MMUC}\}, \rho_{\{a^*_{MMUC}\}}\right), & \text{if } p > \Gamma, \\ \left(\emptyset, \rho_{\{\emptyset\}}\right), & \text{if } p < \Gamma, \end{cases}$$

with the minimum mean uniform cost (MMUC) location estimate

$$a^*_{MMUC} = \operatorname*{argmin}_{a \in \mathcal{X}} \int \mathbf{1}_{r_0}(a,x)\, s(x)dx\,,$$

the threshold Γ is a function of fixed costs and $s(\cdot)$ as specified next for specific applications along with optimal expected losses.

APPLICATION-SPECIFIC BERNOULLI ESTIMATORS

Estimator	c_{00}	c_{01}	c_{10}	c_{11}	c	Tolerance parameter, r_0
1. MPE	0	1	1	0	1	Max. tolerable error
2. MMOL	0	c_A (asset value)	c_M (missile cost)	c_M	c_A	Lethal range

1. **Minimum probability of error (MPE) estimator.** Generalizes the MPE detector [2], and integrates the detection error with the error in localisation beyond maximum tolerable distance

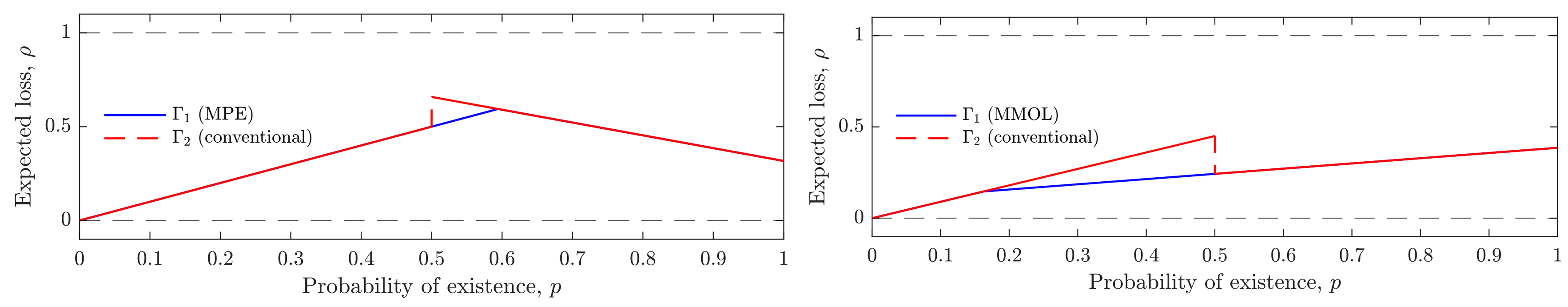
$$\Gamma = \left[2 - \int \mathbf{1}_{r_0}(a^*_{MMUC}, x)s(x)dx\right]^{-1},$$
$$\rho_{\{a^*_{MMUC}\}} = 1 - p + p \cdot \int \mathbf{1}_{r_0}(a^*_{MMUC}, x)s(x)dx,$$
$$\rho_{\{\emptyset\}} = p.$$

2. **Minimum mean operational loss (MMOL) estimator.** Generalizes classical umbrella problem [3], and models a decision about committing a missile with the lethal range r_0 and cost c_M to defend an asset c_A against a Bernoulli target

$$\Gamma = \frac{c_M}{c_A}\left[1 - \int \mathbf{1}_{r_0}(a^*_{MMUC}, x)s(x)dx\right]^{-1},$$
$$\rho_{\{a^*_{MMUC}\}} = c_M + p \cdot c_A \cdot \int \mathbf{1}_{r_0}(a^*_{MMUC}, x)s(x)dx,$$
$$\rho_{\{\emptyset\}} = p \cdot c_A.$$

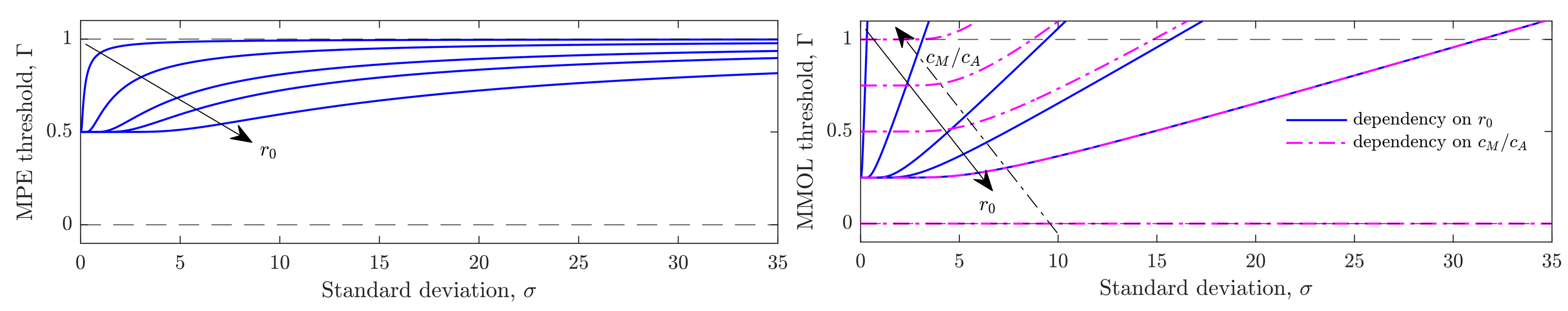
SIMULATIONS WITH A BERNOULLI-GAUSSIAN TARGET

COMPARISON AGAINST THE CONVENTIONAL ESTIMATOR



Reporting a Bernoulli target with $s(\cdot) = \mathcal{N}(\cdot; 0, 1)$ and $r_0 = 1$ using optimal and conventional thresholds. Depending on the application, Γ_1 is higher (MPE) or lower (MMOL, $c_A = 0.9$, $c_M = 0.1$) than conventional $\Gamma_2 = 0.5$ to yield minimum expected loss.

THRESHOLD AS A FUNCTION OF SPATIAL UNCERTAINTY



The growth of spatial uncertainty ultimately leads to the higher optimal threshold values. Naturally, the MPE threshold is never below 0.5. The MMOL threshold can go below this value as the cost c_M decreases with respect to cost c_A.

CONCLUSION

We proposed an approach to producing Bernoulli point estimates that are both optimal (i.e., minimise the expected loss value) and tailored to specific practical applications. Future work will be concerned with extending this approach to multi-Bernoulli filters.

1. Ristic, B., Vo, B.T., Vo, B.N. and Farina, A., 2013. A tutorial on Bernoulli filters: theory, implementation and applications. *IEEE Transactions on Signal Processing*, *61*(13), pp.3406-3430.
2. Middleton, D., 2012. *Non-Gaussian statistical communication theory* (Vol. 22). John Wiley & Sons.
3. Winkler, R. L., 2003. *An Introduction to Bayesian Inference and Decision*. United States: Probabilistic Pub.

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